Computation of Homology

MATHEMATICS SENIOR SEMINAR

Author: Hunter Hasenfus
Advisor: Tibor Beke

University of Massachusetts-Lowell
2023

Computation of Homology

Mathematics Senior Seminar

Hunter Hasenfus

Abstract

Topology's objective is to understand and formalize shape, whatever that may be. Despite its initial, seemingly miraculous departure from geometry—replacing distance metrics with open and closed sets—topology maintains a profound connection to it. Through a series of morphisms, a topology on a set can be transformed into simplicial sets, then into a chain of connected abelian groups. These groups reveal invariant characteristics of the topology, known as Betti numbers, which are crucial in comprehending the shape of mathematical objects. These algebraic manipulations allow for the mathematical objects to retain structure throughout the process. The paper's objective is to elucidate the computation of homology from various perspectives, demonstrating its application in both traditional topological settings and extended data structures. This exploration underlines both the elegance embedded within the abstract nature of homology in addition to the significance of topological methods in comprehending complex shapes.

Introduction

To understand homology, the notion of topology must be extended into combinatorial structures known as simplicial complexes. This allows for a continuous abstract representation of a space to be represented in discrete structures with concrete points and connections. It turns out there are different grains of abstraction from which this process can be understood from, each offering its unique insights and challenges.

The study of Homology allows for further identification of topological objects, uncovering the underlying shape and essence of spaces that might otherwise elude understanding. The process delineates the shape of the space with certain invariant characteristics.

We will begin with preliminary knowledge in 2.1.2 and then move towards constructs that allow for ease of computation in 2.1.2. After words, we will dive into constructs that allow for ease of theoretical expressibility in 3.1, and then finally exemplify homology's utility in 4.2.

Preliminary

2.1 From Point-Set to Combinatorial Topology

The conceptual viewpoint of point-set topology allows for a forumulaic continuous nature. The notions are abstracted from the concepts of open and closed sets which holster definitions that cater towards analytical perspectives.

2.1.1 Topological Spaces

To begin, we assume conceptual understanding of logic and set theory. Topology is founded on the ideas of inclusion and exclusion; this can be illustrated by means of set theory or geometry.

Definition 1. Topology A *topology* on a set X is a collection T of subsets of X having the following properties:

1. $\emptyset$ and X are in T.
2. The union of the elements of any subcollection of T is in T.
3. The intersection of the elements of any finite subcollection of T is in T.

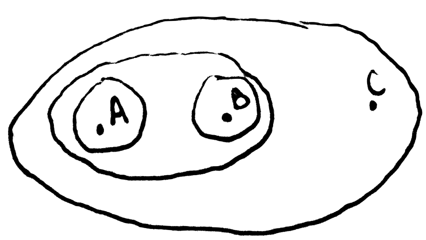


Figure 2.1: $X = \{A, B, C\}, T = \{\{A, B, C\}, \{A, B\}, \{A\}, \{B\}\}$.

Following very cleanly from this are the ideas of openness and closedness.

Definition 2. (Open-set) Let X be a set equipped with a topology $\mathcal{T}$, then $S \in \mathcal{T}$ means that S is *open*. The complement of an open set is a *closed set*.

The example in Figure 2.1, illustrates an instance whereby the set-theoretic axioms of a topology hold. Of great importance in topology is the concept of continuous functions. Burgeoning from the analytical study of $\mathbb{R}$, an analytical perspective of ϵ, δ is met with mathematically equivalent representations using open/closed sets.

Definition 3. (Continuity) A function $f : X \to Y$ is *continuous* if for every open set A in Y, the pre-image $f^{-1}(A)$ is open in X.

The fundamental comparison between topological spaces X and Y is,

Definition 4. (Homeomorphism) A homeomorphism $f : X \to Y$ is a 1-1 onto function, such that both f and f^{-1} are continuous. We say that X is homeomorphic to Y, denoted $X \approx Y$, and that X and Y have the same topological type.

To facilitate understanding of homology, it would help to have examples illustrated from the perspective of 3-dimensional surfaces or 2-manifolds. Homeomorphisms between topological spaces allow for the morphing of one space into another. This can enable topological spaces to be studied from the lens of category theory; having objects and morphisms between objects. It turns out these point-set representations can be shown to have equivalence to a more combinatorial representation.

2.1.2 Simplicial Complexes

Different objectives require differing frameworks or viewpoints on the same problem. Point-set topology enables the concepts of infinite sets to be well understood. To reach a more computational methodology there must be geometric definitions that allow us to talk about combinations and there must be objects we want to combine. Said objects of interest are simplices, these manifestations when combined can be equivalent to an underlying space and allow for the ease of computation. An interesting analog here is how the motivation of point-set topology expressed in the form of metric spaces and epsilon balls is ditched for higher abstraction. We will begin with the geometric intuitions and then abstract away.

The constituent part of a simplicial complex is a k-dimensional simplex:

Definition 5. (k-simplex) A k-simplex is the convex hull of $k + 1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the vertices of the simplex.

Integral to the notion of simplex is that of a face, much like a triangle is composed of 3 edges, and each edge is composed of two vertices.

Definition 6. (face) Let σ be a k-simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$. A simplex τ defined by $T \subseteq S$ is a face of σ.

These ideas motivate the larger structure that is a collection of such simplices:

Definition 7. (geometric simplicial complex) A simplicial complex K is a finite set of simplices such that:

1. Every face of a simplex in K is in K, and

2. The non-empty intersection of any two simplices of K is a face of each of them.

As mentioned at the start of the section, we are merely interested in the elements and their subset inclusion to exclusion. Therefore as a means of raising abstraction, we will define the abstract version of the aforementioned definition. Given a geometric simplicial complex, the vertices and faces give rise to an abstract simplicial complex.

Definition 8. (Abstract simplicial complex) *An abstract simplicial complex* K consists of a set S of finite sets such that if $A \in S$, so is every subset of A. We say $A \in S$ is an (abstract) k-simplex of dimension k if $|A| = k + 1$.

To illustrate the connections to point-set topology consider the simplicial complex:

$$\mathcal{K} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{c, d\}, \{b, d\}, \{a, b, c\}\}.$$

This definition shows that the inclusion and exclusion of separate entities within the complex are fundamental. Allowing the capacity to understand the relation to a point set topology or the intersection and union of subsets. Imparting the geometric notion into the above definition we attain,

Definition 9. (Vertex scheme) Let K be a simplicial complex with vertices V and let S be the collection of all subsets $\{v_0, v_1, \ldots, v_k\}$ of V such that the vertices $v_0, v_1, \ldots, v_k$ span a simplex of K. The collection S is called the *vertex scheme* of K.

Previously simplicial complex K was defined solely in terms of subsets of K. Defining the vertex set V, $V(\mathcal{K}) = \{a, b, c, d\}$. With this labelling along with the definition of a vertex scheme, the geometric understanding of d-dimensional simplices becomes understandable. Every abstract simplicial complex is the vertex scheme of infinitely many geometric simplicial complexes; two sample geometric realizations in Figure 2.4 are each valid for the abstract simplicial complex K.

Figure 2.2 shows the geometric interpretation of the simplicial complex K with labels. Figure 2.3 is provided to bridge the relationship between point-set topology and simplicial complexes. The linkages between layers allow for an understanding of *face* and corresponding higher simplex.

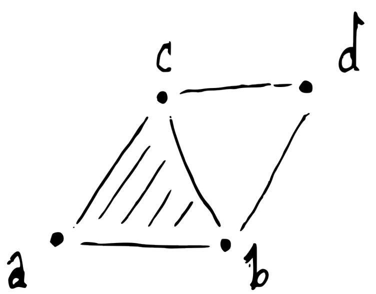


Figure 2.2: Geometric interpretation of K

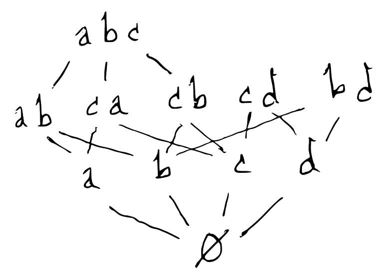


Figure 2.3: View of K through poset visualization.

Vertex schemes allow the specifics of spanning sets, combinations, and geometric realizations. Within the jump from abstract simplicial complex to vertex scheme, the concept of combination has varying properties of choice that construct wholy different spaces.

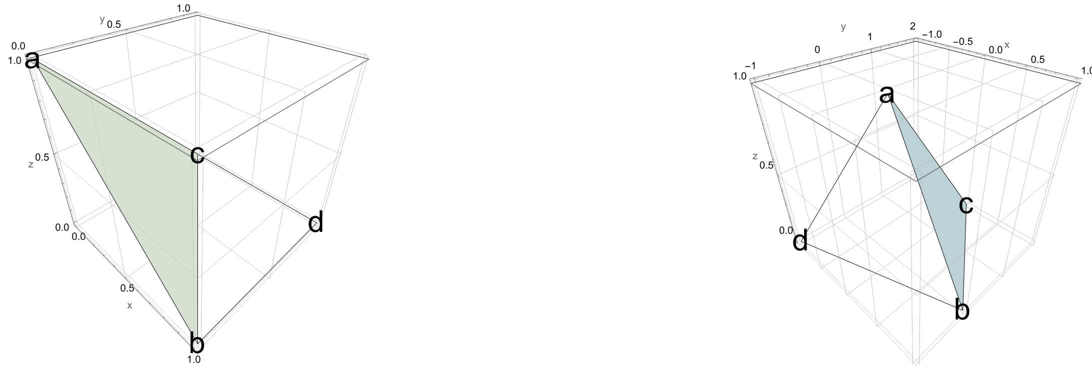
Definition 10. (Combination) Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{V}^d$. A *linear combination* is $x = \sum_{i=0}^{k} \lambda_i p_i$, for some $\lambda_i \in \mathbb{F}$. An *affine combination* is a linear combination with $\sum_{i=0}^{k} \lambda_i = 1$ with $\lambda_i \in \mathbb{R}$. A *convex combination* is an affine combination with $\lambda_i \geq 0$, for all i. The set of all convex combinations is the convex hull.

To connect each of the concepts defined above is a notion of equivalence or isomorphism; a full relationship exists between the abstract simplicial complex and the vertex schemes.

Definition 11. (Isomorphism) Let K_1, K_2 be abstract simplicial complexes with vertices V_1, V_2 and subset collections S_1, S_2, respectively. An *isomorphism* between K_1, K_2 is a bijection $\phi : V_1 \to V_2$, such that the sets in S_1 and S_2 are the same under the renaming of the vertices by ϕ and its inverse.

Theorem 3.2 Every abstract complex S is isomorphic to the vertex scheme of some simplicial complex K. Two simplicial complexes are isomorphic iff their vertex schemes are isomorphic as abstract simplicial complexes.

In Figure 2.4, the abstract simplicial complexes K_1 and K_2 have respective vertex schemes. Given there exists a bijection that maintains the set denominations between the abstract simpli-



(a) $|K_1|$ (b) $|K_2|$

Figure 2.4: Geometric realizations

cial complexes, they are isomorphic. The connection between topological spaces and simplicial complexes resides in the notion of triangulation. A way of carving up the soft and squishy surfaces into a vertex scheme that maintains the original topological nature.

Definition 12. (Triangulation) A *triangulation* of a topological space X is a homeomorphism $h : |K| \to X$, where K is a simplicial complex.

Here $|K|$ represents the geometric realization of abstract simplicial complex K. Triangulation connects back to the point-set notions of manifolds and surfaces and presents equivalence towards the combinatorial representations.

Categorical Perspective

3.1 Simplical Homology to Singular Homology

Homology was originally presented from the standpoint of simplicial homology using the lens of simplicial complexes. This standpoint was initially fruitful, however through time, flaws in its expressability led to unintelligible complications. The non-finite but countable framing on the problem through simplicial sets allowed for a machinery of connectivity that wasn't present in the simplicial complexes.

Building off the concept of simplicial complex, to understand homology and how it is achieved it very much helps to extend this idea to a different framing known as simplicial sets. This allows for a way to think about simplicial complex K as a sequence of sets $X_0, X_1, X_2, \ldots, X_n$, where K $= X_0 \cup X_1 \cup X_2 \cup ... \cup X_n$. The index of each sequenced set denotes the degree of simplices, thus X_0 holds the vertices of the simplicial complex.

Definition 13. Semisimplicial Set A simplicial set X is a sequence of sets $X_0, X_1, X_2, \ldots$, and functions $d_0, d_1 : X_1 \to X_0$, $d_0, d_1, d_2 : X_2 \to X_1$, and so on (in general, we have $n + 1$ functions from $X_n \to X_{n-1}$ for every $n \geq 1$), such that the simplicial identities are satisfied:

$$d_i d_j = d_{j-1} d_i$$

whenever $i < j$ and those equations make sense.

Now with this notation, we can compute the homology of a simplicial set. We are interested in the free abelian groups generated by all X_i. A free abelian group satisfies its 'abelian' and 'group' nature, and is a set S equipped with a binary operation + that is both a group and is communitative under the operation. The 'free' property arises from the existence of a basis, where the entirety of the group can be generated from a finite sum of a set with coefficients. If we denote the n-simplices as X_n, then n-th abelian group is $S_n(K)$. With this idea, we can then describe the boundary operator. It has a strong relationship to the functions defined on the simplicial set.

Homology deals with the change between immediate dimensional simplices (0-dim - 1-dim, 1-dim - 2-dim and so on.) To formalize this we use the boundary operation:

Definition 14. Boundary Operators For all $n \geq 1$, the boundary operators

$$\partial_n : S_n(X) \to S_{n-1}(X)$$

are defined by sending $\sigma \in X_n$ to

$$\sum_{k=0}^{n} (-1)^k d_k \sigma.$$

We also define $\partial_0 : S_0(X) \to 0$ to be the zero homomorphism.

With the notion of mapping between levels of simplices, we can understand homology. It is the quotient space between two underlying spaces or sets. Given set X_i, provided $i > 0$, with the boundary operation, the domain $S_i(X)$ can be split into two subspaces. The first space is the elements mapped to the 0 element, known as the kernel of ∂. The second space is the elements mapped to unique representations within S_{i-1}. Denoting $Z_i(X)$ as $ker(\partial_i)$ and $B_i(X)$ as $im(\partial_{i+1})$.

These subspaces or sub-abelian groups have important intuitive notions. $Z_i(X)$ is conceptually understood as the set of i-dimensional cycles because in 10, the boundary of a cycle equates to 0. $B_i(X)$ is understood as i-dimensional boundaries as they represent the boundary of a higher dimensional simplex. Homology in essence takes the quotient of these subspaces:

$$H_i(X) = ker(\partial_i)/im(\partial_{i+1}) = Z_i(X)/B_i(X)$$

Homology looks at the parts within dimensions that are holes. To understand the overall process of homology's computation from a topological space, I will use the concepts from category theory to illustrate the transition.

Definition 15. Category A Category consists of the following:

- A class ob(C) of objects in C.
- $\forall$ pair of objects $X, Y \in ob(C)$, a set of morphisms denoted $Hom_C(X, Y)$.
- $\forall X \in ob(C)$, an identity morphism $1_X \in Hom_C(X, X)$
- $\forall (X, Y, Z) \in ob(C)$ a composition operation $Hom_C(X, Y) \times Hom_C(Y, Z) \to Hom_C(X, Z)$, written as $g\dot{f}$. Where the composition must satisfy identity and associativity.

The utility of expressing this process within this framing of Category Theory lies in the jumping between different mathematical structures. Beginning with some soft squishy topological space, the goal is to extract the dimensionality of certain abelian groups. To get from one to the other, we must have preservation of important information and we must have awareness of the structure of each category of objects.

$$\text{Top} \longrightarrow \text{Simplicial Sets} \longrightarrow \text{Abelian groups}$$

- The category Set where ob(Set) is the class of all sets, and morphisms are functions between sets.
- The category Top where ob(Top) is the class of all topological spaces, and morphisms are continuous functions between such spaces.
- The category Ab were ob(Ab) is the class of all free abelian groups, and morphisms are group homomorphisms between abelian groups.

Beginning, much like the simplicial complex approach is a way to represent the topological spaces, here we are considering simplicial sets. Topological spaces, specifically n-dimensional manifolds reside in Euclidean space; through this the relationship between geometric and topological objects spawns.

Definition 16. N-simplex For any $n \geq 0$, the standard n-simplex Δ_n is a subspace of $\mathbb{R}^{n+1}$, defined as the convex hull of the standard basis $\{e_0, e_1, \ldots, e_n\}$. In other words,

$$\Delta_n = \left\{ \sum_i t_i e_i : \sum t_i = 1, t_i \geq 0 \right\}$$

A very visualizable method, comparable to the notion of an abstract simplicial complex having a geometric realization. Here we couple this idea with both a topological space and a continuum:

Definition 17. Singular Simplices Let X be a topological space. Define $\text{Sing}_n(X)$ to be the set of all continuous maps $\Delta_n \to X$.

Now instead of thinking about a finite set of simplices, we conceptualize a continuum of sets (or levels) whereby the sets contain at each n-level all continuous maps from a given n-simplex to the topological space X. Earlier within 13 there needed to be a function d_i that mapped n-simplices to the lower level. Here d can be defined with respect to the geometric underpinning of said topological space:

$$d_i : Sing_n(X) \to Sing_{n-1}(X)$$

This allows for the conveyance of topological space into a simplicial set. A way of translating from apples to oranges. The idea is to understand how to go from Topological spaces to Simplicial sets. This necessitates the following:

Definition 18. Functor A *functor* $F : \mathcal{C} \to \mathcal{D}$ of categories consists of

- An assignment $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ from objects to objects, and
- for all $X, Y \in \mathrm{ob}(\mathcal{C})$, there is a function $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.

Furthermore, we must have $F(1_X) = 1_{F(X)}$ for all $X \in \mathrm{ob}(\mathcal{C})$, and for all composable pairs of morphisms $f, g \in \mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

Now $Sing_n$ can be understood as a functor $Top \to Set$, and S_n discussed earlier can be thought of as $Top \to Ab$.

In the event of a continuous function between topological spaces, there exists both a canonical function between sets by applying $Sing_n$ and a canonical function between abelian groups by applying S_n. By canonical, I mean unique.

Thus its clear that since the codomain of $Sing_n$ is Set, defining a function $Free : Set \to Ab$ could be composed with $Sing_n$ to yield S_n. $S_n : Top \to Ab$ is a composition of the two functors $Free \circ Sing_n$.

It is apparent that within the concept of functor, one can think of the simplicial set as a sequence of functions itself. Then what does $Sing$ really do? It is a functor to a collection of functors; a collection of functors must be understood as a category. The definition of simplicial sets is intrinsically linked to the degeneracy maps d_i. We need to further break down $Sing(X)$ constituent parts into categorical representations to give way to the functorial view of simplicial sets- putting them together into the simplicial set construction. The domain of such functors is presumed to be n-simplices, here we define it further:

Definition 19. Let Δ_{inj} denote the category with objects $\text{ob}(\Delta_{\text{inj}}) = \{[0],[1],[2],\ldots\}$, and morphisms between objects given by

$$\text{Hom}_{\Delta_{\text{inj}}}([a],[b]) = \{\text{injective functions } f : \{0,1,\ldots,a\} \to \{0,1,\ldots,b\} \text{ that preserve order}\}.$$

A key aspect of this construction is the properties of injectivity and order preservation. Much like an infinite intersection of finite sets yields a finite set, the composition of a series of injective functions preserves injectivity. The composition of order-preserving functions also preserves order. This gets very deeply at the concept of degeneracy maps, however, the direction of such arrows are pointed the wrong way, thus necessitating the following definition:

Definition 20. Opposite Category Let C be a category. Then the opposite category C^{op} is a category such that $\text{ob}(C^{\text{op}}) = \text{ob}(C)$, but for any $X, Y \in \text{ob}(C^{\text{op}})$, we define

$$\text{Hom}_{C^{\text{op}}}(X,Y) = \text{Hom}_C(Y,X).$$

If $f \in \text{Hom}_C(Y,X)$ is a morphism, we denote the corresponding morphism in $\text{Hom}_{C^{\text{op}}}$ by f^{op}. The composition law we use is

$$(f \circ g)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}}.$$

Elucidating the motivation some more,

objects	domain	image
Δ_{inj}	0,1	0,1, 0,2, 1,2
$\Delta_{\text{inj}}^{\text{op}}$	0,1, 0,2, 1,2	0,1

Table 3.1: Example of regular and opposite category.

The designated functor for simplicial sets that was imagined before can be represented as a category $\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, Set)$ with $\text{ob}(\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, Set))$ as all functors between the designated categories and $\text{Hom}_{\text{Fun}(\Delta^n, Set)}(C,D)$ are the natural transformations between the functors, the d_is.

Definition 21. Natural-transformation Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\Theta : F \to G$ consists of maps $\Theta_X : F(X) \to G(X)$ for all $X \in \text{ob}(\mathcal{C})$, such that for all maps $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{CD} F(X) @>{\Theta_X}>> G(X) \\ @V{F(f)}VV @VV{G(f)}V \\ F(Y) @>>{\Theta_Y}> G(Y) \end{CD}$$

In this view, provided with a continuous map between topological spaces X and Y (homomorphism) we can understand the established category of functors with its natural transformations as:

$$\begin{CD} \mathrm{Sing}_n(X) @>{d_i}>> \mathrm{Sing}_{n-1}(X) \\ @V{Sing_n(f)}VV @VV{Sing_{n-1}(f)}V \\ \mathrm{Sing}_n(Y) @>>{d_i}> \mathrm{Sing}_{n-1}(Y) \end{CD}$$

Now we have:

$$\mathrm{Sing}_n(X) = \mathrm{Top} \to \mathrm{Fun}(\Delta_{\mathrm{inj}}, \mathrm{Set})$$

With a functor to simplicial sets, there exists a functor from simplicial sets to chain complexes. There then exists a slew of functors that morph chain complexes of abelian groups into abelian groups. These abelian groups are then put into a quotient relation to compute the homology groups.

A chain complex is as the name precludes, a chain of complexes. The construction requires a combination of a novel category akin to Δ_{inj} and Ab.

- Let Fil denote the category with one object for each non-negative integer, no morphisms from a to b if a < b, an and a unique morphism otherwise.

It is evident from the definition how it relates to $\Delta_{\mathrm{inj}}^{\mathrm{op}}$. Each element and its subsequent element have morphisms that spawn a downstream cascade due to its compositional nature. With this we can define,

Definition 22. Chain Complex Letting functor Fil $\to$ Ab be a sequence of abelian groups with boundary maps (∂_i) between them. A *chain complex* of abelian groups is such a functor with the property:

$$\partial_{i-1} \circ \partial_i = 0 \quad \forall i \geq 2$$

A similar transformation takes place with chain complexes as they did with simplicial sets; the construction defines a category chAb where ob(chAb) are the defined functors above, and the morphisms are the natural transformations between the functors. It is apparent that the puzzle pieces fit,

$$
\begin{array}{ccccccc}
\mathrm{Sing}_0(X) & \xleftarrow[d_1]{} & \mathrm{Sing}_1(X) & \xleftarrow[d_2]{} & \mathrm{Sing}_1(X) & \xleftarrow[d_i]{} & \ldots \\
{\scriptstyle Free}\downarrow & & \downarrow{\scriptstyle Free} & & \downarrow{\scriptstyle Free} & & \downarrow{\scriptstyle Free} \\
\mathrm{S}_0(X) & \xleftarrow[\partial_1]{} & \mathrm{S}_1(X) & \xleftarrow[\partial_2]{} & \mathrm{S}_2(X) & \xleftarrow[\partial_i]{} & \ldots
\end{array}
$$

This can be understood as a functor mapping simplical sets to chain complexes. Define such functor

$$S_* : \mathrm{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{chAb}$$

Thus finishing off the assembly, all three idealized abelian groups, the end product H_n, and its quotient relation constituents Z_n and B_n are realized as functors,

$$Z_n, B_n, H_n : \mathrm{chAb} \to \mathrm{Ab}$$

Where they are plucked from the chain complex:

$$
\begin{array}{ccccccc}
\emptyset & & \mathrm{B}_0(X) & & \mathrm{B}_1(X) & & \ldots \\
{\scriptstyle im(\partial_0)}\uparrow & & {\scriptstyle im(\partial_1)}\uparrow & & {\scriptstyle im(\partial_2)}\uparrow & & {\scriptstyle im(\partial_i)}\uparrow \\
\mathrm{S}_0(X) & \xleftarrow[\partial_1]{} & \mathrm{S}_1(X) & \xleftarrow[\partial_2 i]{} & \mathrm{S}_2(X) & \xleftarrow[\partial_i]{} & \ldots \\
\downarrow{\scriptstyle ker(\partial_0)} & & \downarrow{\scriptstyle ker(\partial_1)} & & \downarrow{\scriptstyle ker(\partial_2)} & & \downarrow{\scriptstyle ker(\partial_i)} \\
\mathrm{Z}_0(X) & & \mathrm{Z}_1(X) & & \mathrm{Z}_2(X) & & \ldots
\end{array}
$$

With such machinery defined, the homology group functor H_n can be illustrated as a composition of the following functors.

$$H_n : \text{Topological Space} \xrightarrow{Sing} \text{Simplicial Set} \xrightarrow{S_*} \text{Chain Complex} \xrightarrow{H_n} \text{Abelian Group}$$

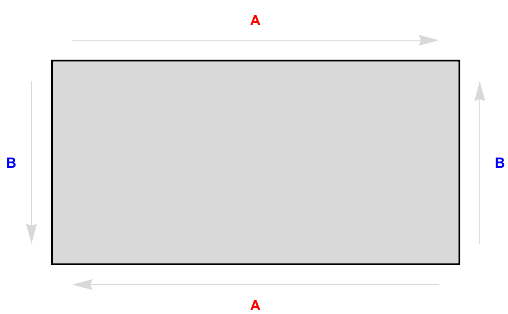
The elegance of this culmination is purely theoretical as the nature of functors and categorical insights do not lend utility in the realm of computation. For the construction of simplicial homology does this so seamlessly described in finite discrete structures.

Matrix Representation and Vector space analog

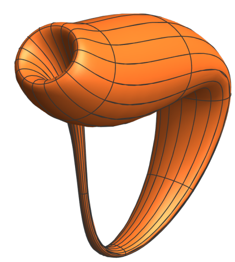
4.1 Computation of Homology for low dimensional surfaces

Now with the computational constructs of simplicial complexes and the conceptual machinery from singular homology, I will then express the analogs towards vector spaces and the resulting computation of such objects.

Firstly the process of computing homology will be followed out on the Klein Bottle. The following is the planar diagram of a Klein Bottle as well as its 3-D rendering:



(a) Klein Bottle Planar Diagram



(b) 3-D Rendering of Klein Bottle

Shown below is a minimal triangulation of the Klein bottle. From left to right the progression is from 0-dim simplices to 2-dim simplices. The inspiration from the triangulation came from [?].

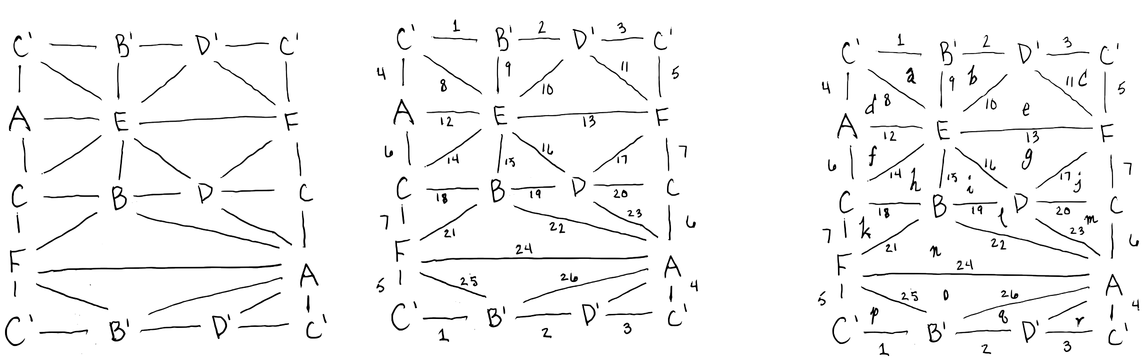


Figure 4.2: Klein Bottle Triangulation

These higher dimension simplices can be represented in terms of their faces using the equipped boundary operator. Previously denoted as,

$$\partial_p : C_p(K) \rightarrow C_{p-1}(K)$$

Within the calculation of the Klein bottle, we have 18 2-dim simplices (Triangles) meaning $C_p(K)$ would be represented as a 18-dimensional vector, and 27 1-dim simplices (edges) equating to C_{p-1} being a 27-dimensional vector. Thus the boundary operation would be a 27 x 18 dimensional matrix.Consider two 0-1 vectors of dimension 6, $\mathbf{v}_1$ and $\mathbf{v}_2$.

An important aspect of the computation of homology is the notion of coefficients. These coefficients arise in the jump from a set to a free abelian group or in this instance the vector space equivalents. Homology is computed using the dimensionality of certain subspaces of the boundary operator and this requires an understanding of how the elements within the group or vector space can be combined. These coefficients can be any ring including $\mathcal{Z}/2, \mathcal{Z}, \mathcal{Q}$, and many others. The choice of coefficients plays a large role in the interpretation of the Betti numbers. Here we have chosen $\mathcal{Z}/2$ as it allows for the simplest understanding of cycles.

Adding these vectors with mod 2 coefficients gives us a vector of all zeros due to the cancellation at the second position:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now consider two different 0-1 vectors. Adding these vectors with mod 2 coefficients gives us a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_4$:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathbf{v}_3 + \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The n-dimensional simplicial complexes can have their boundary operation equivalents translated into 0-1 vectors to fill a matrix. This then allows for the matrix to be treated as a vector space with mod 2 coefficients. After the matrices are row-reduced to echelon form, the coinciding homological information can be deduced.

Definition. For any $p \in \{0, 1, 2, \ldots\}$, the *pth homology* of a simplicial complex K is the following quotient vector space and subsequently its dimension known as its Betti Number:

$$H_p(K) := \frac{\ker(\partial_p)}{\operatorname{im}(\partial_{p+1})} \quad \beta_p(K) := \dim H_p(K) = \dim \ker(\partial_p) - \dim \operatorname{im}(\partial_{p+1})$$

Using linear algebra, the corresponding matrices with mod-2 field coefficients are row-reduced into the following matrices.

Δ^1	$\partial(\Delta^1)$
1	$C' + B'$
2	$B' + D'$
3	$D' + C'$
4	$C' + A$
5	$C' + F$
6	$A + C$
7	$C + F$
8	$C' + E$
9	$B' + E$
10	$E + D'$
11	$D' + F$
12	$A + E$
13	$E + F$
14	$C + E$
15	$E + B$
16	$E + D$
17	$D + F$
18	$C + B$
19	$B + D$
20	$D + C$
21	$B + F$
22	$B + A$
23	$D + A$
24	$F + A$
25	$F + B'$
26	$A + B'$
27	$A + D'$

Figure 4.3: 1-simlices and their faces

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 4.4: $\partial(\Delta^1)$ matrix in RREF

Δ^2	$\partial(\Delta^2)$
a	$1 + 9 + 8$
b	$2 + 9 + 10$
c	$3 + 11 + 5$
d	$4 + 8 + 12$
e	$10 + 11 + 13$
f	$6 + 12 + 14$
g	$13 + 16 + 17$
h	$14 + 18 + 15$
i	$15 + 16 + 19$
j	$17 + 7 + 20$
k	$7 + 18 + 21$
l	$19 + 22 + 23$
m	$20 + 23 + 6$
n	$21 + 22 + 24$
o	$24 + 25 + 26$
p	$5 + 25 + 1$
q	$26 + 2 + 27$
r	$27 + 3 + 4$

Figure 4.5: 2-simplices and their faces

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 4.6: $\partial(\Delta^2)$ matrix in RREF

An equivalent way to represent this from the matrices above:

- ker(∂_p) are the linearly dependent vectors of the respective boundary operation, therefor $\dim \ker(\partial_p)$ is the number of zeroed out rows.

- im(∂_{p+1} are the linearly independent vectors of the higher dimensions respective boundary operation, therefor $\dim \operatorname{im}(\partial_{p+1})$ is the number of non-zero rows.

Therefore, proceeding with the Klein Bottle:

$$\beta_2(K) := N_2 - R_3 = 1 - 0 = 1$$

$$\beta_1(K) := N_1 - R_2 = 19 - 17 = 2$$

$$\beta_0(K) := N_0 - R_1 = 9 - 8 = 1$$

4.2 Homology of similar shapes

Lastly, I would like to illustrate the capacity for classification and identification that is inherent within homology. Using two shapes that appear very similar, we will triangulate them, and then provide the mod 2 homology. The first shape is a double torus, or in essence a connected sum of two tori. The second shape is a block with two spheres taken out, reminscent of a block of cheese.Below are 3-D renderings of each.

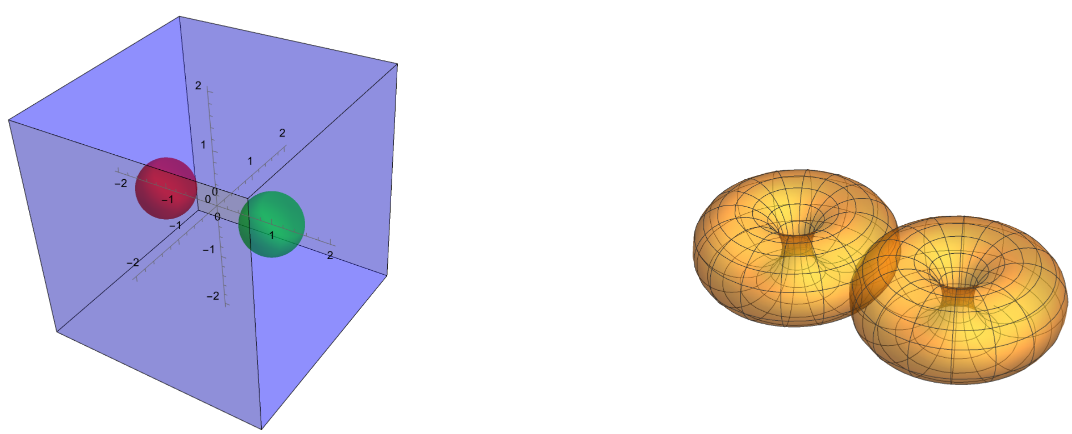


Figure 4.7: 3-D Rendering

The concept of a connected sum in topology can be more intuitively understood through its geometric representation rather than solely by its algebraic formulation. It involves taking two topological spaces and creating a new space by a specific 'cut and glue' procedure. Concretely, this procedure entails removing a 2-simplex (a disk-like region) from each space and then identifying (or gluing) their corresponding lower-dimensional boundary simplices (the circular boundaries of the removed disks) in a pairwise fashion. The resulting space, termed the connected sum of the original spaces, effectively merges their topological features while maintaining the essential properties of each.

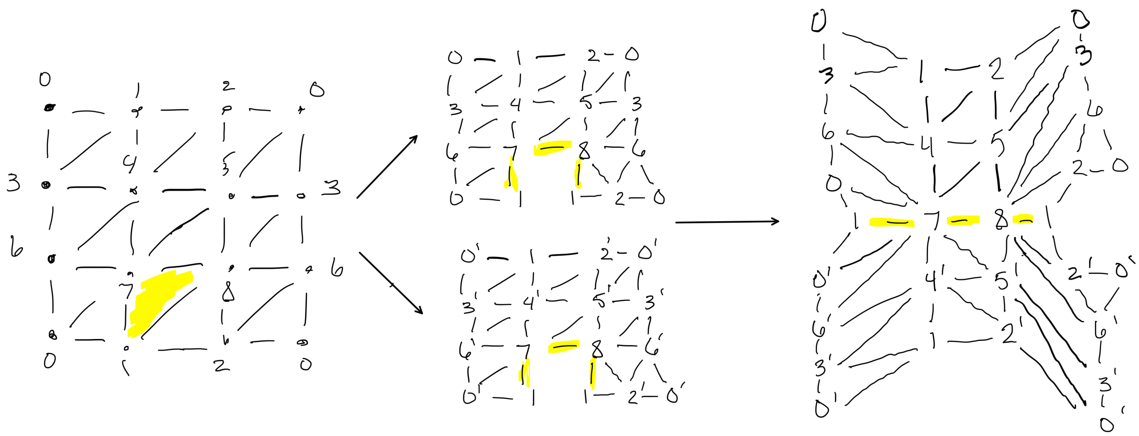


Figure 4.8: Double Torus

The torus by nature contains a void within it, it is not solid; this holds for the double torus. The block of cheese instead is solid, it contains within it higher-dimensional simplices or tetrahedron. The process of triangulating the block of cheese began with a triangulation of a block and the removal of two inner blocks for the holes within.

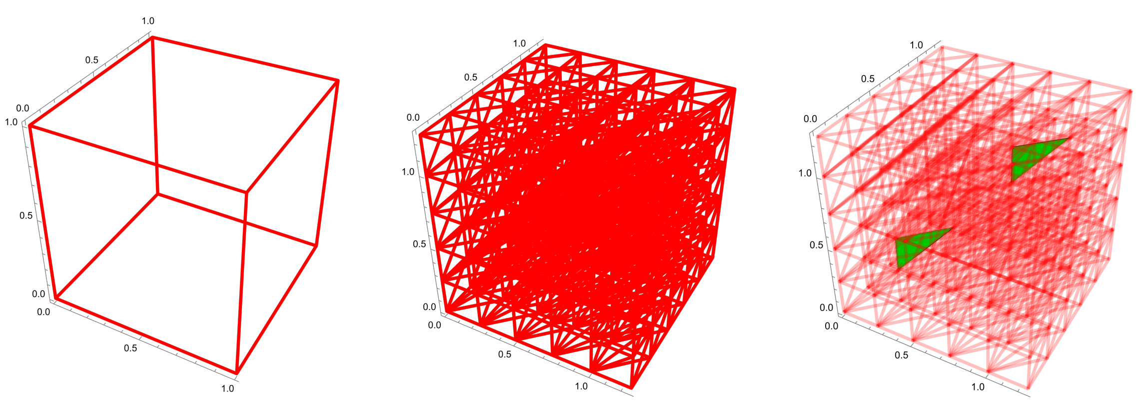


Figure 4.9: Cheese Block Triangulation

After the same process used with the Klein Bottle, consisting of labeling all 0-2 simplices, converting to boundary matrices, and computing the homology of each, there exists a clear distinction. The double torus has 34 2-simplices, 51 1-simplices, and 15 0-simplces. After converting into matrices and row reducing, the resulting homology is:

$$\beta_2(\mathrm{T}) := 1 \quad \beta_1(T) := 4 \quad \beta_0(\mathrm{T}) := 1$$

This has a very intuitive understanding, the double torus is a connective sum of two tori thus the void in a single tori is extended to a larger yet singular void. There now exist two visual holes in the space, within the single torus the 1st Betti number was 2 now it's 4. The external continuity remains singular within the process of the connective sum thus the 0th Betti number of 1 remains.

As for the homology of the block of cheese, the calculation is a bit more tricky and interesting. There now exists consideration of tetrahedrons, thus plucking out the holes in the cheese requires eliminating higher-level simplices (tetrahedrons).

Within the triangulation of the cheese block in Figure 4.9, I have visualized the tetrahedrons that are removed on the inside. Within the triangulation, I spared to provide the matrices as the dimensionality of the space is enormous; the amounts are: 3-simplices :: 750, 2-simplices :: 1650, 1-simplices :: 1115, and 0-simplices :: 216. To provide some relative framing, the Betti numbers of the sphere (S) and the solid sphere (SS) precede the Betti numbers for the block of cheese:

$$\beta_2(S) := 1 \quad \beta_2(SS) = 0 \quad \beta_2(BC) = 2$$

$$\beta_1(S) := 0 \quad \beta_1(SS) = 0 \quad \beta_1(BC) = 0$$

$$\beta_0(S) := 1 \quad \beta_0(SS) = 1 \quad \beta_0(BC) = 1$$

Thus the distinguishing factor between the sphere and solid sphere is the absence of filling space within the sphere which culminates in a $\beta_2 = 1$ or a void. Therefore by triangulating the space for the higher level simplex (tetrahedron), they can be factored into the Betti number 2 to show fullness. What is quite interesting is that the formulation of the double torus was acquired through a connected sum, and it seems that the block of cheese has the equivalent Betti numbers to a connected sum of two spheres. The homology of the two shapes has been understood and each shape can be distinguished to some degree by their invariant properties.

Conclusion

The computation of homology from the perspective of three different angles, the simplicial complexes and their abelian groups, the singular homology process, and the analogous approach through vector spaces. The degree of abstraction that ones seeks to learn the concepts lends to its own fruits; simplicial homology and its analogous translation into vector spaces is fruitful for computation whereas the categoric viewpoint of singular homology aids in logical fluidity.

Bibliography

[1] Jeremy Hanh. Algebraic topology i. Stanford Web, 2021. Accessed on: 2023-11-13.

[2] A. Hatcher. *Algebraic Topology*. Cambridge Univ. Press, Cambridge, UK, 2001.

[3] J. R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, Reading, MA, 1984.

[4] Nina Otter, Mason A Porter, Ulrike Tillmann, Peter Grindrod, and Heather A Harrington. A roadmap for the computation of persistent homology. *EPJ Data Science*, 6(1), aug 2017.

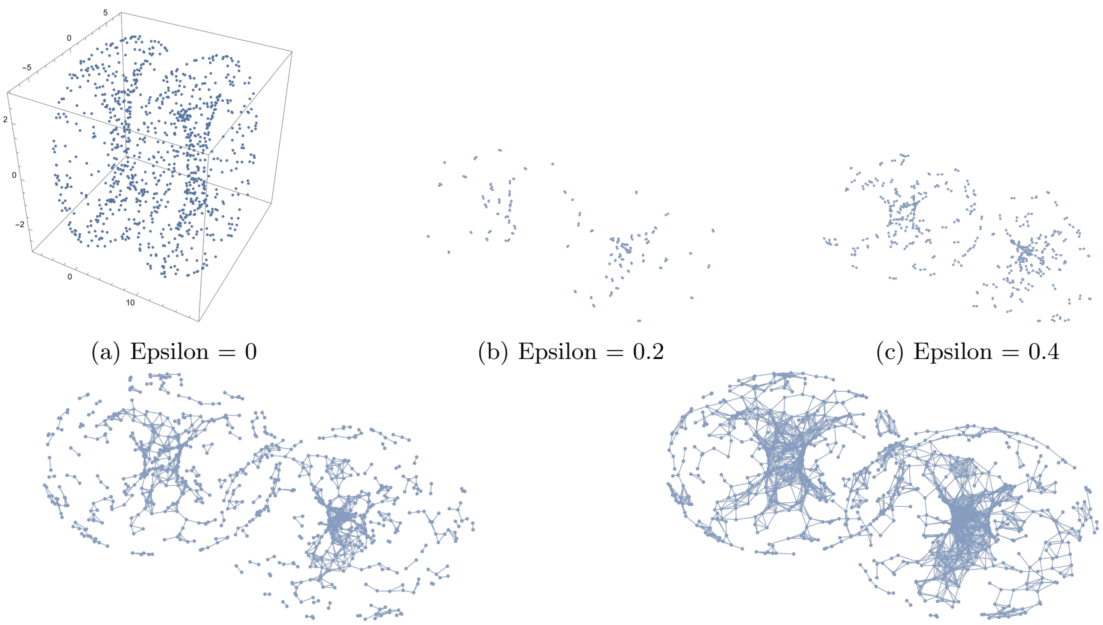
[5] Afra Zomorodian. Introduction to computational topology. Stanford Graphics, 2004. Accessed on: 2023-11-13.

Appendix

Homology will be extended into the realm of $\mathbb{R}^N$ through the concepts of filtration and persistence.

A.1 Association of Simplicial Complexes to Point Clouds

In this section, we explore how point clouds, which are sets of points in a metric space, can be associated with simplicial complexes. This association is fundamental in topological data analysis, as it allows us to study the shape of the data by examining the topological properties of the corresponding simplicial complex.



(a) Epsilon = 0 (b) Epsilon = 0.2 (c) Epsilon = 0.4

(d) Epsilon = 0.8 (e) Epsilon = 1.2

Figure A.1: Sampling of Double Torus

A.1.1 Introduction to Point Clouds

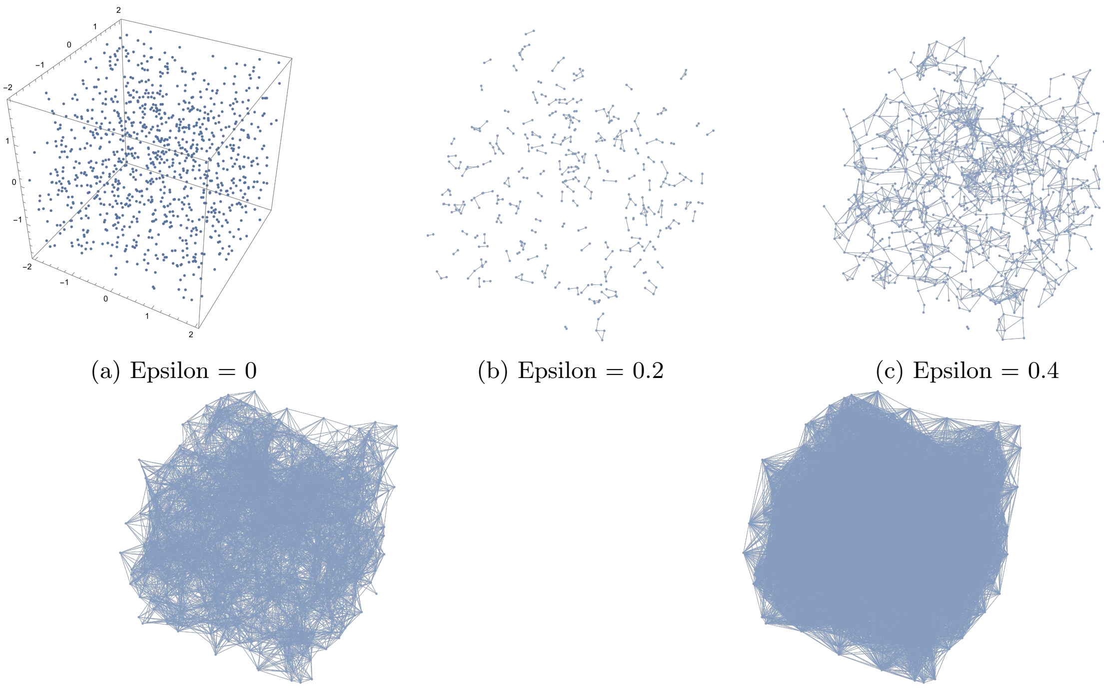
A point cloud is a set of points in some coordinate space. These points can represent a variety of things depending on the context, such as positions of objects in space, data points from a sampling of a manifold, or features from a high-dimensional data set.

A.1.2 Constructing Simplicial Complexes from Point Clouds

To analyze the underlying topological structure of a point cloud, we can construct a simplicial complex that captures the relationships between points. This is often done using techniques such as the Vietoris-Rips or Čech complexes, which create simplices based on proximity or connectivity criteria among points. We will use Vietoris-Rips for the examples. A very illustrative example comes from Robert Grist's "BARCODES: THE PERSISTENT TOPOLOGY OF DATA" [**?**].

A.2 Filtrations on Homology of Simplicial Complexes

Filtrations are a powerful tool in persistent homology, allowing us to study the homology of simplicial complexes at various scales. By applying a filtration to a simplicial complex, we can observe how its homological features evolve as we progressively add simplices based on a certain parameter.



(a) Epsilon = 0 (b) Epsilon = 0.2 (c) Epsilon = 0.4

(d) Epsilon = 0.8 (e) Epsilon = 1.2

Figure A.2: Sampling of Cheese Cube

The beauty of the approach lies in the functionality of homology. Earlier we built upon the categorical foundations, and touched upon the concept of functor 18. This plays a significant role in the utility of persistent homology.